PERGAMON

International Journal of Heat and Mass Transfer 42 (1999) 3539–3544

International Journal of HEAT and MASS TRANSFER

Technical Note

Similarity solutions of stationary thermoelasticity with the frictional heating

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Received 24 July 1998; received in revised form 2 December 1998

Abstract

Similarity transformations are constructed and used to obtain an exact solution for the axisymmetric boundary value problem of an elastic half-space subjected to a continuous point heat source on its surface. It is found to agree with known results, obtained by Parcus with the help of the thermoelastic potential method. The temperature, displacements and stresses are used in the solution of the thermoelasticity contact problem with frictional heating for the 'pin-on–half-space' tribosystem.

1. Introduction and statement of the problem

Similarity methods and their applications to various boundary value problems in engineering have been developed by Hansen [1] Ibragimov [2], Bluman and Cole [3] and Ovsjannikov et al. [4].

In this paper, similarity analysis is applied to the axisymmetric static boundary value problem for a semi-infinite body which is subjected to a cylindrical pin normal to its surface at origin. The pin is embedded into the surface of the half-space by normal load P and rotates uniformly with speed ω. Owing to friction the heat is generated within the contact area. The following assumptions were made for simplicity:

1. the pin is rigid and thermoinsulated;
2. the surface of the half-space outside the region of heating is free and insulated;
3. the power Q of the heat source on the interface is equal to the power of the friction force

$$Q = f\omega aP; \tag{1}$$

4. the system is in steady-state.

The problem under consideration is the sum of two separate problems. The first, when the thermal effects are absent (the pin is stationary, that is, $Q=0$), reduces to a semi-infinite solid, subjected to a concentrated normal force on its boundary. The solution of this problem by means similarity methods was obtained by Chowdhury and Glockner [5] for the 'hot pin' contact problem. We calculate the temperature, displacements and stresses in the second case, when the continuous point heat source of the power Q on the surface of the half-space is acting. The closed form of the solution is derived and is found to agree with a known result, given by Parcus [6].

We note that the axisymmetrical contact problem of stationary thermoelasticity involving frictional heating due to rotating of the sphere on the surface of the half-space has been studied by Barber [7] using the original numerical method and by Yevtushenko and Kulchytsky-Zhyhailo [8,9] by means of the Hankel integral transform method. The corresponding contact problem in which a parabolic annular punch pressed

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0017-9310/99/$ - see front matter
PII: S0017-9310(98)00376-7

Nomenclature

a	radius of the pin
b, c	material constants
E	Young's modulus
f	coefficient of friction
K	thermal conductivity
P	normal force
Q	power of the frictional point heat source
r	radial coordinate
T	temperature
u	radial displacement
w	normal displacement
z	normal coordinate.

Greek symbols

α	coefficient of the thermal expansion
$\delta(\cdot)$	Dirac's delta-function
λ, μ	Lamé constants
ν	Poisson's ratio
ω	rotating speed.

into a plane surface and rotates uniformly is considered by Yevtushenko and Kulchytsky-Zhyhailo [10].

2. Basic equations and boundary conditions

Let us consider a point heat source Q which is localized on the surface of a half-space. Due to axial symmetry, the cylindrical coordinate system r, φ and z is fixed to the source with the z-axis directed into the half-space. The resulting thermal field $T=T(r, z)$ is derived from the following heat conduction problem:

$$\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{\partial^2 T}{\partial z^2}=0 \quad \text{for } r\geq 0, \quad z>0 \tag{2}$$

$$K\frac{\partial T}{\partial z}=-Q\frac{\delta(r)}{r} \quad \text{for } z=0, \quad r\geq 0$$
$$\text{or} \quad \lim_{z\to 0}\int_0^\infty r\frac{\partial T(r,z)}{\partial z}\,\mathrm{d}r=-\frac{Q}{2\pi K} \tag{3}$$

$$T\to 0 \quad \text{as } r, z\to\infty. \tag{4}$$

The Duhamel–Navier equations for static thermoelasticity are [11]

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}-\frac{u}{r^2}+b\frac{\partial^2 u}{\partial z^2}+(1-b)\frac{\partial^2 w}{\partial r\,\partial z}=c\frac{\partial T}{\partial r} \tag{5}$$

$$(1-b)\left(\frac{\partial^2 u}{\partial r\,\partial z}+\frac{1}{r}\frac{\partial u}{\partial z}\right)+b\left(\frac{\partial^2 w}{\partial r^2}+\frac{1}{r}\frac{\partial w}{\partial r}\right)+\frac{\partial^2 w}{\partial z^2}=c\frac{\partial T}{\partial z} \tag{6}$$

where

$$b=\frac{\mu}{\lambda+2\mu}, \quad c=\frac{(1+\nu)\alpha}{1-\nu}, \quad \mu=\frac{E}{2(1+\nu)}, \quad \lambda=\frac{\nu E}{(1+\nu)(1-2\nu)}. \tag{7}$$

The stress–displacement-temperature relations are

$$\sigma_{rr}=\frac{\mu}{b}\left[\frac{\partial u}{\partial r}+(1-2b)\left(\frac{u}{r}+\frac{\partial w}{\partial z}\right)-cT\right] \tag{8}$$

$$\sigma_{\varphi\varphi}=\frac{\mu}{b}\left[(1-2b)\left(\frac{\partial u}{\partial r}+\frac{\partial w}{\partial z}\right)+\frac{u}{r}-cT\right] \tag{9}$$

$$\sigma_{zz}=\frac{\mu}{b}\left[(1-2b)\left(\frac{\partial u}{\partial r}+\frac{u}{r}\right)+\frac{\partial w}{\partial z}-cT\right] \tag{10}$$

$$\sigma_{rz}=\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\right). \tag{11}$$

The boundary conditions on the surface $z=0$ of the half-space are

$$\sigma_{zz}(r,0)=0 \tag{12}$$

$$\sigma_{rz}(r,0)=0. \tag{13}$$

Due to the symmetry of the problem the radial stress vanishes on the axis of the symmetry $r=0$:

$$\lim_{r\to 0} r^2\sigma_{rr}(r,z)=0 \quad \text{for any } z\geq 0. \tag{14}$$

The physical condition

$$\lim_{r\to 0} w(r,z)=\infty \quad \text{for any } z\geq 0 \tag{15}$$

states that the vertical displacement w will be unbounded for any horizontal plane $z=$const at large distances from the source [6].

3. The similarity transformations

We seek a transformation of the system of partial differential equations (2), (5) and (6) and boundary conditions (3), (4) and (12)–(15), which will reduce the number of the independent variables r and z by one, i.e. lead to ordinary differential equations (ODEs). To that end we select a one-parameter transformation group, i.e. recast the problem in terms of the following dimensionless variables

$$r^*=\frac{r}{r_0},\quad z^*=\frac{z}{z_0},\quad T^*=\frac{T}{T_0},\quad u^*=\frac{u}{u_0},\quad w^*=\frac{w}{w_0} \tag{16}$$

where r_0 and z_0 are arbitrary reference variables and T_0, u_0 and w_0 depend on the boundary conditions.

Taking into account the definition (16), Eqs. (2)–(6) and (12)–(15) can be rewritten as

$$\frac{\partial^2 T^*}{\partial r^{*2}}+\frac{1}{r^*}\frac{\partial T^*}{\partial r^*}+\left[\frac{r_0^2}{z_0^2}\right]\frac{\partial^2 T^*}{\partial z^{*2}}=0 \quad \text{for } r^*\geq 0,\ z^*>0 \tag{17}$$

$$\frac{\partial T^*}{\partial z^*}=-\frac{Q}{2\pi K}\left[\frac{z_0}{r_0}\frac{1}{T_0r_0}\right]\frac{\delta(r^*)}{r^*} \quad \text{for } z^*=0,\ r^*\geq 0 \tag{18}$$

$$T^*\to 0 \quad \text{at } r^*, z^*\to\infty \tag{19}$$

$$\frac{\partial^2 u^*}{\partial r^{*2}}+\frac{1}{r^*}\frac{\partial u^*}{\partial r^*}-\frac{u^*}{r^{*2}}+\left[\frac{r_0^2}{z_0^2}\right]b\frac{\partial^2 u^*}{\partial z^{*2}}+\left[\frac{r_0}{z_0}\frac{w_0}{u_0}\right](1-b)\frac{\partial^2 w^*}{\partial r^*\partial z^*}=\left[\frac{T_0r_0}{u_0}\right]c\frac{\partial T^*}{\partial r^*} \tag{20}$$

$$(1-b)\left(\frac{\partial^2 u^*}{\partial r^*\partial z^*}+\frac{1}{r^*}\frac{\partial u^*}{\partial z^*}\right)+b\left[\frac{z_0}{r_0}\frac{w_0}{u_0}\right]\left(\frac{\partial^2 w^*}{\partial r^{*2}}+\frac{1}{r^*}\frac{\partial w^*}{\partial r^*}\right)+\left[\frac{r_0}{z_0}\frac{w_0}{u_0}\right]\frac{\partial^2 w^*}{\partial z^{*2}}=\left[\frac{T_0r_0}{u_0}\right]c\frac{\partial T^*}{\partial z^*} \tag{21}$$

$$\sigma_{zz}(r^*,0)=\frac{\mu}{b}\left\{(1-2b)\left(\frac{\partial u^*}{\partial r^*}+\frac{u^*}{r^*}\right)+\left[\frac{r_0}{z_0}\frac{w_0}{u_0}\right]\frac{\partial w^*}{\partial z^*}-\left[\frac{T_0r_0}{u_0}\right]cT^*\right\}\Bigg|_{z^*=0}=0 \tag{22}$$

$$\sigma_{rz}(r^*,0)=\mu\left(\frac{\partial u^*}{\partial z^*}+\left[\frac{z_0}{r_0}\frac{w_0}{u_0}\right]\frac{\partial w^*}{\partial r^*}\right)\Bigg|_{z^*=0}=0 \tag{23}$$

$$\lim_{r^*\to 0} r^{*2}\sigma_{rr}(r^*,z^*)={}^{*2}\,\frac{\mu}{b}\left\{\frac{\partial u^*}{\partial r^*}+(1-2b)\left(\frac{u^*}{r^*}+\left[\frac{r_0}{z_0}\frac{w_0}{u_0}\right]\frac{\partial w^*}{\partial z^*}\right)-\left[\frac{T_0r_0}{u_0}\right]cT^*\right\}\Bigg|_{r^*=0}=0 \tag{24}$$

$$\lim_{r^*\to 0} w(r^*,z^*)=\infty \tag{25}$$

where the relation $\delta(r^*r_0)=\delta(r^*)/r_0$ has been used.

Furthermore, we will find an absolute invariant η which is a function of the independent variables r and z alone. Absolute invariance demands that each of the factors enclosed in the rectangular brackets in Eqs. (17)–(25) should be equal to one. This gives

$$\frac{r_0}{z_0}=1,\quad \frac{w_0}{u_0}=1,\quad \frac{T_0r_0}{u_0}=1,\quad T_0r_0=1. \tag{26}$$

Eqs. (26) suggest the following form for the similarity transformation:

$$T(r,z)=\frac{1}{r}\theta(\eta),\quad u(r,z)=U(\eta),\quad w(r,z)=W(\eta) \tag{27}$$

where the new independent variable is the absolute invariant

$$\eta=\frac{z}{r}. \tag{28}$$

Differentiating of the relations (27) with respect to r and z gives

$$\frac{\partial T}{\partial r}=-\frac{1}{r^2}\left(\theta+\eta\frac{\mathrm{d}\theta}{\mathrm{d}\eta}\right),\quad \frac{\partial T}{\partial z}=\frac{1}{r^2}\frac{\mathrm{d}\theta}{\mathrm{d}\eta}$$

$$\frac{\mathrm{d}^2T}{\mathrm{d}r^2}=\frac{1}{r^3}\left(2\theta+4\eta\frac{\mathrm{d}\theta}{\mathrm{d}\eta}+\eta^2\frac{\mathrm{d}^2\theta}{\mathrm{d}\eta^2}\right),\quad \frac{\partial^2T}{\partial z^2}=\frac{1}{r^3}\frac{\mathrm{d}^2\theta}{\mathrm{d}\eta^2}\tag{29}$$

and

$$\frac{\partial u}{\partial r}=-\frac{1}{r}\eta\frac{\mathrm{d}U}{\mathrm{d}\eta},\quad > \frac{\partial u}{\partial z}=\frac{1}{r}\frac{\mathrm{d}U}{\mathrm{d}\eta},$$

$$\frac{\partial^2 u}{\partial r^2}=\frac{1}{r^2}\left(2\eta\frac{\mathrm{d}U}{\mathrm{d}\eta}+\eta^2\frac{\mathrm{d}^2U}{\mathrm{d}\eta^2}\right)$$

$$\frac{\partial^2 u}{\partial z^2}=\frac{1}{r^2}\frac{\mathrm{d}^2U}{\mathrm{d}\eta^2},\quad \frac{\partial^2 u}{\partial r\,\partial z}=-\frac{1}{r^2}\left(\frac{\mathrm{d}U}{\mathrm{d}\eta}+\eta\frac{\mathrm{d}^2U}{\mathrm{d}\eta^2}\right).\tag{30}$$

4. Integration of the ODEs

Substituting the relations (27)–(30) into Eqs. (2), (5) and (6) yields

$$(1+\eta^2)\frac{\mathrm{d}^2\theta}{\mathrm{d}\eta^2}+3\eta\frac{\mathrm{d}\theta}{\mathrm{d}\eta}+\theta=0\tag{31}$$

$$(b+\eta^2)\frac{\mathrm{d}^2U}{\mathrm{d}\eta^2}+\eta\frac{\mathrm{d}U}{\mathrm{d}\eta}-U-(1-b)\left(\eta\frac{\mathrm{d}^2W}{\mathrm{d}\eta^2}+\frac{\mathrm{d}W}{\mathrm{d}\eta}\right)=-c\left(\eta\frac{\mathrm{d}\theta}{\mathrm{d}\eta}+\theta\right)\tag{32}$$

$$-(1-b)\eta\frac{\mathrm{d}^2U}{\mathrm{d}\eta^2}+b\left(\eta^2\frac{\mathrm{d}^2W}{\mathrm{d}\eta^2}+\eta\frac{\mathrm{d}W}{\mathrm{d}\eta}\right)+\frac{\mathrm{d}^2W}{\mathrm{d}\eta^2}=c\frac{\mathrm{d}\theta}{\mathrm{d}\eta}.\tag{33}$$

Thus, we obtain three coupled ordinary differential equations (31)–(33) with variable coefficients for the unknown functions θ, U and W. Eqs. (31) and (32) can be integrated directly. We find

$$(1+\eta^2)\frac{\mathrm{d}\theta}{\mathrm{d}\eta}+\eta\theta=D_1\tag{34}$$

$$(b+\eta^2)\frac{\mathrm{d}U}{\mathrm{d}\eta}-\eta U-(1-b)\eta\frac{\mathrm{d}W}{\mathrm{d}\eta}=-c\eta\theta+D_2.\tag{35}$$

We multiply Eq. (32) by η and subtract the result from Eq. (33). After dividing by $(1+\eta^2)^{3/2}$ and integrating we find

$$-\eta\frac{\mathrm{d}U}{\mathrm{d}\eta}+U+\frac{\mathrm{d}W}{\mathrm{d}\eta}=c\theta+\frac{D_3}{(1+\eta^2)^{1/2}}.\tag{36}$$

Dividing Eq. (34) by $(1+\eta^2)^{1/2}$ and integrating gives

$$\theta(\eta)=\frac{D_1}{(1+\eta^2)^{1/2}}\ln[\eta+(1+\eta^2)^{1/2}]+\frac{D_4}{(1+\eta^2)^{1/2}}.\tag{37}$$

Making use of the boundary conditions (3) and (4), it is found that

$$D_1=0,\quad D_4=\frac{Q}{2\pi K}.\tag{38}$$

Eliminating $\mathrm{d}W/\mathrm{d}\eta$ from Eqs. (35) and (36) leads to

$$b(1+\eta^2)\frac{\mathrm{d}U}{\mathrm{d}\eta}-b\eta U=D_2+[(1-b)D_3-bcD_4]\frac{\eta}{(1+\eta^2)^{1/2}}.\tag{39}$$

Dividing Eq. (39) by $(1+\eta^2)^{3/2}$ and integrating we obtained

$$bU(\eta)=\eta D_2-[(1-b)D_3-bcD_4]\frac{1}{(1+\eta^2)^{1/2}}+(1+\eta^2)^{1/2}D_5.\tag{40}$$

Substituting the function U (40) into Eq. (36) and integrating yields

$$bW(\eta)=(D_3-D_5)\ln[\eta+(1+\eta^2)^{1/2}]-[(1-b)D_3-bcD_4]\frac{\eta}{2(1+\eta^2)^{1/2}}+D_6.\tag{41}$$

In Eqs. (40) and (41) D_i, $i=2, 3, 5, 6$ are arbitrary constants, which are determined from the boundary conditions (12)–(15). The condition (15) implies that $D_6=(D_5-D_3)\ln r$.

Introducing Eqs. (27), (40) and (41) into Eq. (8) gives

$$\sigma_{rr}(r,z)=\frac{\mu}{b}\left\{-2D_2\frac{z}{r^2}-2D_5\frac{(r^2+z^2)^{1/2}}{r^2}+(2D_5-bD_3-bcD_4)\frac{1}{(r^2+z^2)^{1/2}}+\frac{[(1-b)D_3-bcD_4]r^2}{(r^2+z^2)^{3/2}}\right\}\tag{42}$$

and from the symmetry condition (14) we find that

$$D_2=-D_5.\tag{43}$$

Taking the relation (43) into account, Eqs. (40) and (42) can be rewritten as

$$bU(\eta) = \frac{D_5}{\eta + (1+\eta^2)^{1/2}} - [(1-b)D_3 - bcD_4]\frac{1}{2(1+\eta^2)^{1/2}} \tag{44}$$

$$\sigma_{rr}(r,z) = \frac{\mu}{b}\left\{ -\frac{2D_5}{z+(r^2+z^2)^{1/2}} + (2D_5 - bD_3 - bcD_4)\frac{1}{(r^2+z^2)^{1/2}} + [(1-b)D_3 - bcD_4)]\frac{r^2}{(r^2+z^2)^{3/2}} \right\}. \tag{45}$$

Writing Eqs. (9)–(11) in terms of the functions U (44) and W (41) and taking the relations (27) into account we find

$$\sigma_{\varphi\varphi}(r,z) = \frac{\mu}{b}\left\{ -\frac{2D_5 z}{(r^2+z^2)^{1/2}[z+(r^2+z^2)^{1/2}]} + \frac{2D_5 - bD_3 - bcD_4}{(r^2+z^2)^{1/2}} \right\} \tag{46}$$

$$\sigma_{zz}(r,z) = \frac{\mu}{b}\left\{ [-2D_5 + (3-2b)D_3 - 2bcD_4]\frac{1}{(r^2+z^2)^{1/2}} - [(1-b)D_3 - bcD_4]\frac{r^2}{(r^2+z^2)^{3/2}} \right\} \tag{47}$$

$$\sigma_{rz}(r,z) = \frac{\mu}{b}\left\{ \frac{(D_3 - 2D_5)r}{(r^2+z^2)^{1/2}[z+(r^2+z^2)^{1/2}]} + \frac{[(1-b)D_3 - bcD_4]rz}{(r^2+z^2)^{3/2}} \right\}. \tag{48}$$

Substituting the expressions (47) and (48) into the boundary conditions (12) and (13) leads to a system of two algebraic equations for the constants D_3 and D_5:

$$-2D_5 + (2-b)D_3 - bcD_4 = 0$$

$$D_3 - 2D_5 = 0. \tag{49}$$

From these equations we find

$$D_3 = \frac{bc}{(1-b)}\frac{Q}{2\pi K}, \quad D_5 = \frac{D_3}{2}. \tag{50}$$

5. The final result

Using (7), the coefficient $c/(1-b)$ in Eqs. (50) is equal to

$$\frac{c}{1-b} = 2(1+\nu)\alpha \tag{51}$$

and Eqs. (27), (37), (41) and (44)–(48) can be expressed in the final form as

$$T(r,z) = \frac{Q}{2\pi K}\frac{1}{R} \tag{52}$$

$$u(r,z) = (1+\nu)\alpha\frac{Q}{2\pi K}\frac{r}{z+R}, \quad w(r,z) = (1+\nu)\alpha\frac{Q}{2\pi K}\ln(z+R) \tag{53}$$

$$\sigma_{rr}(r,z) = -2\mu(1+\nu)\alpha\frac{Q}{2\pi K}\frac{r}{z+R} \tag{54}$$

$$\sigma_{\varphi\varphi}(r,z) = -2\mu(1+\nu)\alpha\frac{Q}{2\pi K}\frac{z}{R(z+R)} \tag{55}$$

$$\sigma_{zz}(r,z) = 0, \quad \sigma_{rz}(r,z) = 0 \tag{56}$$

where $R = (r^2 + z^2)^{1/2}$.

The expressions (52)–(56) agree with those arrived at by Parcus [6] by means of the thermoelastic potential method. Adding this solution to the known solution [5] for the case of concentrated normal force applied to the surface of the half-space, we obtain the solution of the contact problem with frictional heat generation for the tribosystem 'pin-on–half-space'.

Acknowledgements

The work was carried out while A. A. Yevtushenko was a Visiting Faculty Member in The Department of Structural Mechanics at The Technology University of Lodz, Poland.

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